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# ON A THIRD ORDER LINEAR PREDICTOR-CORRECTOR DIGITAL FILTER

JULY 1980

FINAL REPORT





INSTRUMENTATION DIRECTORATE
US ARMY WHITE SANDS MISSILE RANGE
WHITE SANDS MISSILE RANGE, NEW MEXICO 88002

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A third order linear predictor-corrector digital filter with parameters $w_1$, $w_2$, and $w_3$ is introduced and discussed in this paper. Four particular ones, with different choices of $w_1$, $w_2$, and $w_3$, have been or are being used at White Sands Missile Range (WSMR) of the US Army. In particular, McCool's QD filter and Shepherd's third order parabolic spline filter were devised at WSMR for missile trajectory smoothing, reconstruction, and differentiation. Stability conditions for these filters are derived and discussed. In addition, the filter equations are uncoupled so that design criteria can be obtained.

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## 1. INTRODUCTION

This paper is concerned with a class of linear predictor-corrector digital filters of the third order with three parameters. This class includes the four digital filters that have been or are being used at White Sands Missile Range (WSMR) of the United States Army. Two of them, namely the McCool's QD filter [3] and Shepherd's third order parabolic spline filter [4], were devised at WSMR for missile trajectory smoothing, reconstruction and differentiation. All of them have applications in real time radar tracking digital servomechanisms. The theory for this paper was developed largely from internal notes of WSMR by Chui [1] and Shepherd [4] although some laborious detail in [1] and [4] is not given here. The class of linear predictor-corrector digital filters to be discussed here can be defined as in the following. Set

$$\underline{w} = [w_1, w_2, w_3]^T . \tag{1.1}$$

Here, and throughout, the superscript $T$ indicates the transpose of a matrix. Hence, $\underline{w}$ in (1.1) is a three-dimensional constant column vector. If $\{x_i\}$, $i = 0, 1, \ldots,$ denotes an input signal, $h$ a positive constant, and the sequence of three-dimensional vectors

$$\underline{y}_i = [y_i, y_i', y_i'']^T, \tag{1.2}$$

$i = 0, 1, \ldots,$ denotes the output response corresponding to the input signal $\{x_i\}$ and subject to the initial condition $\underline{y}_{-1} = [y_{-1}, y_{-1}', y_{-1}'']^T$, then the class of digital filters we consider is defined by

$$\underline{y}_{p+1} = \underline{u}_p(ph + h) + (x_{p+1} - u_p(ph + h))\underline{w} \tag{1.3}$$

for $p = -1, 0, 1, \ldots,$ where

$$u_p(t) = y_p + y_p'(t - ph) + \frac{1}{2}y_p''(t - ph)^2, \quad \text{and} \tag{1.4}$$
$$\underline{u}_p(t) = [u_p(t), u_p'(t), u_p''(t)]^T .$$

The real-valued function $u_p(t)$ and the vector-valued function $\underline{u}_p(t)$ are both called predictor functions. Also, $\underline{u}_p(ph + h)$ is called the prediction at the time $t = ph + h$ based on the output response $\underline{y}_p$, while $(x_{p+1} - u_p(ph + h))\underline{w}$ is called the correction at the time $t = ph + h$. With $\underline{w} = [\alpha, \beta/h, \gamma/h^2]^T$ this filter appears as the general $\alpha$-$\beta$-$\gamma$ filter (cf. Steelman [5] and the references therein).

It is easy to verify that the filter (1.3), (1.4) is equivalent to the filter

$$\underline{y}_{p+1} = A\underline{y}_p + x_{p+1}\underline{w}, \tag{1.5}$$

$p = -1, 0, 1, \ldots,$ where $A$ is the $3 \times 3$ matrix

$$A = \begin{bmatrix} 1 - w_1 & (1 - w_1)h & \frac{1}{2}(1 - w_1)h^2 \\ -w_2 & 1 - w_2h & h - \frac{1}{2}w_2h^2 \\ -w_3 & -w_3h & 1 - \frac{1}{2}w_3h^2 \end{bmatrix} . \tag{1.6}$$

This matrix formulation exhibits the filter as a third order linear difference equation. With input $\{x_i\}$ where $x_i = 0$ for $i < 0$, output $\{\underline{y}_i\}$, $i = 0, 1, \ldots,$ and initial value $\underline{y}_{-1}$ the general solution can be shown by induction to be

$$\underline{y}_{p+1} = A^{p+2}\underline{y}_{-1} + A^{p+1}x_0\underline{w} + A^px_1\underline{w} + \ldots + Ax_p\underline{w} + x_{p+1}\underline{w}. \tag{1.7}$$

If the response due to arbitrary initial value $\underline{y}_{-1}$ is to damp out, it is apparent from (1.7) that

$$\lim_{n\to\infty} A^n \underline{x} = 0 \quad \text{for all } x \text{ in } \mathbb{R}^3. \tag{1.8}$$

The relationship between this observation and the stability of the filter (1.5), (1.6) is developed in section 2. More precisely, we will show that <u>if all the eigenvalues</u> $\lambda_1, \lambda_2, \lambda_3$ <u>of the matrix</u> A <u>lie in the open unit disc</u> $|z| < 1$, <u>then the digital filter</u> (1.3) - (1.4), <u>or equivalently</u> (1.5) - (1.6), <u>is stable</u>. <u>Furthermore, the condition</u> $|\lambda_j| < 1$, <u>for</u> $j = 1, 2, 3$, <u>is satisfied if and only if the condition</u> (1.8) <u>is satisfied</u>. We will study this eigenvalue problem via z-transforms. This technique will be extended to "uncouple" the system (1.3) - (1.4) solving for each of the output response sequences $\{y_p\}$, $\{y'_p\}$, and $\{y''_p\}$ in terms of the input signal $\{x_p\}$ and the initial conditions $y_{-1}$, $y'_{-1}$, and $y''_{-1}$ respectively in recursive forms. These three recursive formulas will allow the studying of each of the three output response sequences $\{y_p\}$, $\{y'_p\}$, and $\{y''_p\}$ individually without referring to the other two. In particular, design criteria can be formulated.

In section 3, we will consider the special case of Shepherd's "parabolic spline predictor corrector filter", which is obtained from (1.5) and (1.6) by setting

$$\underline{w} = \left[\delta, \frac{2\delta}{h}, \frac{2\delta}{h^2}\right]^T \tag{1.9}$$

where $\delta$ is considered as a design parameter.

In section 4 we consider somewhat more briefly McCool's "QD" filter [3], for which (with $\underline{w} = [\alpha, \beta/h, \gamma/h^2]^T$)

$$[\alpha,\beta,\gamma] = \left[\frac{60M^2}{10M^3 + 33M^2 + 23M - 6}, \beta = \frac{2\alpha}{M}, \gamma = \frac{2\alpha}{M^2}\right],$$

Morrison's "fading memory polynomial filter of degree 2" [5], for which

$$[\alpha,\beta,\gamma] = [1-\theta^3, \tfrac{3}{2}(1-\theta)^2(1+\theta), (1-\theta)^3],$$

and an $\alpha - \beta$ filter (for which one can set $w_3 = 0$ and $y''_{-1} = 0$ in the beginning) studied by Gonzales in 1968 (cf. [2]).

## 2. STABILITY AND UNCOUPLING OF THE FILTERS

We will use the technique of z-transforms to study the class of digital filters defined by (1.3) - (1.4), or equivalently (1.5) - (1.6). If $\{b_j\}$ is a bi-infinite sequence of complex numbers, then the z-transform of the sequence $\{b_j\}$, $j = \ldots, -1, 0, 1, \ldots,$ is the formal Laurent series

$$B(z) = \sum_{j=-\infty}^{\infty} b_j z^{-j} .$$

Let $\{x_p\}$, $\{y_p\}$, $\{y'_p\}$ and $\{y''_p\}$ be defined as in section 1. We will set $x_p = 0$ if $p < 0$, and $y_p = y'_p = y''_p = 0$ if $p < -1$. Hence, the z-transforms of these sequences are given by

$$X = X(z) = \sum_{p=0}^{\infty} x_p z^{-p},$$

$$Y_1 = Y_1(z) = \sum_{p=-1}^{\infty} y_p z^{-p},$$

$$Y_2 = Y_2(z) = \sum_{p=-1}^{\infty} y'_p z^{-p},$$

$$Y_3 = Y_3(z) = \sum_{p=-1}^{\infty} y''_p z^{-p}$$

respectively. If we re-write the matrix system (1.5) in the form of a system of three simultaneous difference equations and take the z-transforms of each of these three equations, we obtain the following system of simultaneous linear algebraic equations

$$\begin{cases} (z-1+w_1)Y_1 - (h-hw_1)Y_2 - \left(\frac{1}{2}h^2 - \frac{1}{2}h^2w_1\right)Y_3 = w_1zX \\ w_2Y_1 + (z-1+hw_2)Y_2 - (h-\frac{1}{2}h^2w_2)Y_3 = w_2zX \\ w_3Y_1 + hw_3Y_2 + (z-1+\frac{1}{2}h^2w_3)Y_3 = w_3zX \ . \end{cases} \tag{2.1}$$

In matrix representation, (2.1) can be written as

$$(A - zI_3)[Y_1, Y_2, Y_3]^T = -zX\underline{w} \tag{2.2}$$

where $I_3$ is the $3 \times 3$ identity matrix. Let $H_1(z)$, $H_2(z)$, and $H_3(z)$ be the transfer functions of this digital filter; that is,

$$Y_j(z) = H_j(z)X(z) \tag{2.3}$$

for $j = 1, 2, 3$. Then $H_1(z)$, $H_2(z)$, and $H_3(z)$ can be obtained by solving the linear system (2.2) using Cramer's rule. Hence, they are rational functions in $z^{-1}$ with the same denominator $\det(A - zI_3)$. This shows that if $\det(A - zI_3) \neq 0$ for all $z$ with $|z^{-1}| \leq 1$ or $|z| \geq 1$, or equivalently all the eigenvalues of $A$ lie inside the unit circle $|z| = 1$, then the filter (1.3) - (1.4) is stable. Let $\lambda_1$, $\lambda_2$, $\lambda_3$ be the (not necessarily distinct) eigenvalues of $A$ and let $\underline{x}_1$, $\underline{x}_2$, $\underline{x}_3$ be three corresponding linearly independent eigenvectors. Then for any $\underline{x} \in \mathbb{R}^3$, $\underline{x} = \alpha_1\underline{x}_1 + \alpha_2\underline{x}_2 + \alpha_3\underline{x}_3$ for some constants $\alpha_1$, $\alpha_2$, and $\alpha_3$. Hence, for any positive integer $n$, we have

$$A^n\underline{x} = \alpha_1\lambda_1^n\underline{x}_1 + \alpha_2\lambda_2^n\underline{x}_2 + \alpha_3\lambda_3^n\underline{x}_3 \ .$$

This shows that if $|\lambda_j| < 1$ for $j = 1, 2, 3$, then $A^n\underline{x} \to \underline{0}$ as $n \to \infty$. Conversely, if $A^n\underline{x} \to \underline{0}$ for all $\underline{x} \in \mathbb{R}^3$, we can pick $\underline{x} = \underline{x}_j$ $(j = 1, 2, 3)$ to yield $\lambda_j^n\underline{x}_j = A^n\underline{x}_j \to \underline{0}$ as $n \to \infty$. Since $\underline{x}_j \neq 0$, we must have $|\lambda_1|, |\lambda_2|, |\lambda_3| < 1$. That is, we have established the following

THEOREM 2.1. Let $\lambda_1, \lambda_2, \lambda_3$ be the eigenvalues of the matrix $A$ given in (1.6) and $\Lambda = \max(|\lambda_1|, |\lambda_2|, |\lambda_3|)$. Then the digital filter (1.3) - (1.4) is stable provided $\Lambda < 1$. Furthermore this stability condition $\Lambda < 1$ holds if and only if

$$\lim_{n\to\infty} A^n\underline{x} = \underline{0}$$

for all $\underline{x} \in \mathbb{R}^3$.

We next determine the transfer functions $H_1(z)$, $H_2(z)$, and $H_3(z)$. To do this, we first compute $\det(zI_3 - A)$. It is given by

$$\det(zI_3 - A) = z^3 + \{(w_1 - 3) + w_2h + \tfrac{1}{2}w_3h^2\}z^2 + \{(3 - 2w_1) - w_2h + \tfrac{1}{2}w_3h^2\}z + (w_1 - 1) \ . \qquad (2.4)$$

Hence, by Cramer's rule, we obtain

$$Y_1(z) = \frac{zX(z)}{\det(zI_3 - A)} \begin{vmatrix} w_1 & (w_1 - 1)h & \frac{1}{2}(w_1 - 1)h^2 \\ w_2 & z + w_2h - 1 & \frac{1}{2}w_2h^2 - h \\ w_3 & w_3h & z + \frac{1}{2}w_3h^2 - 1 \end{vmatrix} \qquad (2.5)$$

$$= \frac{zX(z)}{\det(zI_3 - A)} \{w_1z^2 + (\tfrac{1}{2}w_3h^2 + w_2h - 2w_1)z + (\tfrac{1}{2}w_3h^2 - w_2h + w_1)\}.$$

Similarly, we obtain

$$Y_2(z) = \frac{zX(z)}{\det(zI_3 - A)} (z-1)(w_2 z + hw_3 - w_2) , \tag{2.6}$$

$$Y_3(z) = \frac{zX(z)}{\det(zI_3 - A)} (z-1)^2 w_3 . \tag{2.7}$$

Hence, the transfer functions $H_1(z)$, $H_2(z)$, and $H_3(z)$ as defined in (2.3) can be written as

$$H_1(z) = \frac{w_1 + \left(\frac{1}{2}w_3h^2 + w_2h - 2w_1\right)z^{-1} + \left(\frac{1}{2}w_3h^2 - w_2h + w_1\right)z^{-2}}{1 + \{(w_1-3) + w_2h + \frac{1}{2}w_3h^2\}z^{-1} + \{(3-2w_1) - w_2h + \frac{1}{2}w_3h^2\}z^{-2} + (w-1)z^{-3}} \tag{2.8}$$

$$H_2(z) = \frac{w_2 + (hw_3 - 2w_2)z^{-1} + (w_2 - hw_3)z^{-2}}{1+\{(w_1-3) + w_2h + \frac{1}{2}w_3h^2\}z^{-1} + \{(3-2w_1) - w_2h + \frac{1}{2}w_3h^2\}z^{-2} + (w_1-1)z^{-3}} \tag{2.9}$$

and

$$H_3(z) = \frac{w_3 - 2w_3z^{-1} + w_3z^{-2}}{1 + \{(w_1-3) + w_2h + \frac{1}{2}w_3h^2\}z^{-1} + \{(3-2w_1) - w_2h + \frac{1}{2}w_3h^2\}z^{-2} + (w_1-1)z^{-3}} \tag{2.10}$$

If we put the expressions (2.8), (2.9), and (2.10) back into (2.3), multiply the denominator of $H_j(z)$ to $Y_j(z)$, and take the inverse z-transforms of each of the expressions for $j = 1, 2,$ and $3$, we obtain the following

THEOREM 2.2. <u>The digital filter given by</u> (1.3) - (1.4) <u>can be written as three uncoupled recursive digital filters:</u>

$$\begin{aligned} y_p = &-\{(w_1-3) + w_2h + \tfrac{1}{2}w_3h^2\}y_{p-1} - \{(3-2w_1) - w_2h + \tfrac{1}{2}w_3h^2\}y_{p-2} \\ &- (w_1-1)y_{p-3} + w_1x_p + \left(\tfrac{1}{2}w_3h^2 + w_2h - 2w_1\right)x_{p-1} + \left(\tfrac{1}{2}w_3h^2 - w_2h + w_1\right)x_{p-2} , \end{aligned} \tag{2.11}$$

$$\begin{aligned} y'_p = &-\{(w_1-3) + w_2h + \tfrac{1}{2}w_3h^2\}y'_{p-1} - \{(3-2w_1) - w_2h + \tfrac{1}{2}w_3h^2\}y'_{p-2} \\ &- (w_1-1)y'_{p-3} + w_2x_p + (hw_3 - 2w_2)x_{p-1} + (w_2 - hw_3)x_{p-2} , \end{aligned} \tag{2.12}$$

and

$$y''_p = -\{(w_1-3)+w_2h+\tfrac{1}{2}w_3h^2\}y''_{p-1} - \{(3-2w_1)-w_2h+\tfrac{1}{2}w_3h^2\}y''_{p-2}$$

$$- (w_1-1)y''_{p-3} + w_3x_p - 2w_3x_{p-1} + w_3x_{p-2}\,, \qquad (2.13)$$

where $p = 0, 1, 2, \ldots$ with initial conditions $y_{-1}$, $y'_{-1}$, and $y''_{-1}$, and where $x_p = 0$ for $p < 0$ and $y_p = y'_p = y''_p = 0$ for $p < -1$.

We now return to study the stability of the filter (1.3) - (1.4) a little closer via the transfer functions $H_1(z)$, $H_2(z)$, and $H_3(z)$. From (2.4), putting $z = 0$, we see that $\det A = 1-w_1$. This says that $\lambda = 0$ is an eigenvalue of $A$ if and only if $w_1 = 1$. If $w_3 = 0$, then the transfer function $H_3(z)$ is identically zero. On the other hand, if $w_3 \neq 0$, then the value $z^{-1} = 1$ is not a zero of $H_1(z)$, while it is a double zero of $H_3(z)$ and at least a simple zero of $H_2(z)$. Hence, in this case, all the eigenvalues $\lambda_1$, $\lambda_2$, $\lambda_3$ of $A$ are essential in the consideration of stability. More precisely, we have the following.

COROLLARY 2.1. Let $w_3 \neq 0$. Then the digital filter (1.3) - (1.4) is stable if and only if $\Lambda := \max(|\lambda_1|, |\lambda_2|, |\lambda_2|) < 1$.

## 3. A PARABOLIC SPLINE PREDICTOR-CORRECTOR DIGITAL FILTER

This filter is the special case $\underline{w} = \left[\delta, \frac{2\delta}{h}, \frac{2\delta}{h^2}\right]^T$. It was devised by Shepherd [4] and has the following property: Set

$$v_p(t) = y_p + y_p'(t-ph) + \frac{1}{2}\left[-\frac{2\delta}{h^2}y_p - \frac{2\delta}{h}y_p' + (1-\delta)y_p'' + \frac{2\delta}{h^2}x_{p+1}\right](t-ph)^2, \quad (3.1)$$

and define $v(t)$ on $[0, \infty)$ by

$$v(t) = v_p(t) \quad \text{for} \quad ph \leq t < ph + h, \quad p = 0, 1, 2, \dots . \quad (3.2)$$

Here, the input signal $\{x_p\}$ and the output response $\underline{y}_p = [y_p, y_p', y_p'']^T$ satisfy the filter relationship (1.3) - (1.4) with $\underline{w} = [\delta, \frac{2\delta}{h}, \frac{2\delta}{h^2}]^T$. Using this relationship, it is easy to verify that the function $v$ defined by (3.2) is indeed a parabolic (or third order $C^1$) spline on $[0, \infty)$ with knots at $\{0, h, 2h, \dots\}$. Since $v(ph) = y_p$ and $v'(ph) = y_p'$, the spline function $v$ together with its derivative $v'$ interpolate the output response $[y_p, y_p']^T$ at the time $t = ph$, $p = 0, 1, \dots$ . We like to think of the input signal $\{x_p\}$ as a noisy measurement of $\{f(ph)\}$ where $f$ is a fairly smooth function to be reconstructed, and the output response $\underline{y}_p = [y_p, y_p', y_p'']^T$ as an approximation to $[f(ph), f'(ph), f''(ph)]^T$. The approximation improves as $\delta$ tends to 1 as can be seen in (3.3) below. Hence, the spline function $v$ can be considered as an approximation to the function $f$ which we wish to reconstruct from the noisy data $\{x_p\}$. This particular filter is a one-parameter digital filter, parametrized by $\delta$. In fact, if we put $\underline{w} = [\delta, \frac{2\delta}{h}, \frac{2\delta}{h^2}]^T$ in (1.3), then (1.3) - (1.4) becomes

$$\begin{cases} y_{p+1} = (1-\delta)y_p + (1-\delta)hy_p' + \frac{1}{2}(1-\delta)h^2y_p'' + x_{p+1}, \\ y_{p+1}' = -\frac{2\delta}{h}y_p + (1-2\delta)y_p' + (1-\delta)hy_p'' + \frac{2\delta}{h}x_{p+1}, \\ y_{p+1}'' = -\frac{2\delta}{h^2}y_p - \frac{2\delta}{h}y_p' + (1-\delta)y_p'' + \frac{2\delta}{h^2}x_{p+1}, \end{cases} \quad (3.3)$$

and in matrix form, we have $\underline{y}_{p+1} = A_\delta \underline{y}_p + x_{p+1}[\delta, \frac{2\delta}{h}, \frac{2\delta}{h^2}]^T$, with

$$A_\delta = \begin{bmatrix} 1-\delta & (1-\delta)h & \frac{1}{2}(1-\delta)h^2 \\ -\frac{2\delta}{h} & 1-2\delta & (1-\delta)h \\ -\frac{2\delta}{h^2} & -\frac{2\delta}{h} & 1-\delta \end{bmatrix} \tag{3.4}$$

If we put $\delta = 1$, then $y_{p+1} = x_{p+1}$ in (3.3). Hence, if $\delta$ is close to 1, the spline function $v$ is indeed an "approximation" to the noisy input data $\{x_p\}$. However, a digital filter must be stable. Therefore, we will discuss how close can $\delta$ approach 1 so that the digital filter defined by (3.3) remains stable. Since $w_3 = 2\delta/h^2 \neq 0$, we conclude from Corollary 2.1 that the digital filter is stable if and only if $\Lambda_\delta := \max(|\lambda_1|, |\lambda_2|, |\lambda_3|) < 1$ where $\lambda_1$, $\lambda_2$, and $\lambda_3$ are the eigenvalues of $A_\delta$. If $z$ is one of the three eigenvalues $\lambda_1$, $\lambda_2$, and $\lambda_3$, then by (2.4), $z$ must satisfy the equation

$$z^3 + (4\delta - 3)z^2 + 3(1-\delta)z - (1-\delta) = 0. \tag{3.5}$$

This equation does not contain the time increment $h$. Hence, $\Lambda_\delta$ is independent of $h$. For $\delta = 1$, the solutions of (3.5) are $0, 0, -1$, so that $\Lambda_1 = 1$. Hence, the filter is not stable if $\delta = 1$. Let

$$\delta := \delta_n = 1 - \frac{1}{n}, \quad n = \pm 2, \pm 3, \ldots \tag{3.6}$$

so that $\delta_n \to 1$. The following tables indicate the stability of the filter for different values of $n$. Note that for $n = 3$, the roots of (3.5) are

$\frac{1}{3}$, i, -i, so that $\Lambda_{\delta_3} = 1$ and the filter is unstable. Indeed, Steelman [5] pointed out that this filter is stable if and only if $2/3 < \delta < 1$ although a proof is not given in [5]. These tables verify the truth of the statement. Note that in Table 1A, for n = 3, $\delta_n$ should be 2/3 instead of .667 and the filter should be unstable as mentioned above.

TABLE 1A. STABILITY VS. MAXIMUM MODULUS OF THE EIGENVALUES OF $A_1$

| N | DELTA = 1 - 1/N | MAXIMAL MOD OF THE ROOT | STABILITY |
|---|---|---|---|
| 2 | 0.500 | 1.1228020 | NO |
| 3 | 0.667 | 0.9999995 | NO |
| 4 | 0.750 | 0.9158811 | YES |
| 5 | 0.800 | 0.8543339 | YES |
| 6 | 0.833 | 0.8067909 | YES |
| 7 | 0.857 | 0.7685731 | YES |
| 8 | 0.875 | 0.7369311 | YES |
| 9 | 0.889 | 0.7101327 | YES |
| 10 | 0.900 | 0.6870247 | YES |
| 11 | 0.909 | 0.6668116 | YES |
| 12 | 0.917 | 0.6489261 | YES |
| 13 | 0.923 | 0.6329178 | YES |
| 14 | 0.929 | 0.6184893 | YES |
| 15 | 0.933 | 0.6054116 | YES |
| 16 | 0.938 | 0.5934409 | YES |
| 17 | 0.941 | 0.5824399 | YES |
| 18 | 0.944 | 0.5722803 | YES |
| 19 | 0.947 | 0.5627684 | YES |
| 20 | 0.950 | 0.5539731 | YES |
| 21 | 0.952 | 0.5458879 | YES |
| 22 | 0.955 | 0.5380396 | YES |
| 23 | 0.957 | 0.5307868 | YES |
| 24 | 0.958 | 0.5239464 | YES |
| 25 | 0.960 | 0.5174800 | YES |
| 26 | 0.962 | 0.5113528 | YES |
| 27 | 0.963 | 0.5055345 | YES |
| 28 | 0.964 | 0.5000002 | YES |
| 29 | 0.966 | 0.5835004 | YES |
| 30 | 0.967 | 0.6170683 | YES |
| 31 | 0.968 | 0.6419301 | YES |
| 32 | 0.969 | 0.6621327 | YES |
| 33 | 0.970 | 0.6793841 | YES |
| 34 | 0.971 | 0.6942258 | YES |
| 35 | 0.971 | 0.7074718 | YES |
| 36 | 0.972 | 0.7193651 | YES |
| 37 | 0.973 | 0.7301440 | YES |

TABLE 1A (Cont)

| N | DELTA = 1 - 1/N | MAXIMAL MOD OF THE ROOT | STABILITY |
|---|---|---|---|
| 38 | 0.974 | 0.7399907 | YES |
| 39 | 0.974 | 0.7490339 | YES |
| 40 | 0.975 | 0.7573915 | YES |
| 41 | 0.976 | 0.7651453 | YES |
| 42 | 0.976 | 0.7723694 | YES |
| 43 | 0.977 | 0.7791147 | YES |
| 44 | 0.977 | 0.7854404 | YES |
| 45 | 0.978 | 0.7913876 | YES |
| 46 | 0.978 | 0.7969885 | YES |
| 47 | 0.979 | 0.8022742 | YES |
| 48 | 0.979 | 0.8072767 | YES |
| 48 | 0.980 | 0.8120122 | YES |
| 50 | 0.980 | 0.8165183 | YES |

TABLE 1B. STABILITY VS. MAXIMUM MODULUS OF THE EIGENVALUES OF $A_1$

| N | DELTA - 1 + 1/N | MAXIMAL MOD OF THE ROOTS | STABILITY |
|---|---|---|---|
| 2 | 1.500 | 3.4733100 | NO |
| 3 | 1.333 | 2.7423110 | NO |
| 4 | 1.250 | 2.3622870 | NO |
| 5 | 1.200 | 2.1263980 | NO |
| 6 | 1.167 | 1.9643850 | NO |
| 7 | 1.143 | 1.8455800 | NO |
| 8 | 1.125 | 1.7543650 | NO |
| 9 | 1.111 | 1.6819070 | NO |
| 10 | 1.100 | 1.6228290 | NO |
| 11 | 1.091 | 1.5736540 | NO |
| 12 | 1.083 | 1.5320190 | NO |
| 13 | 1.077 | 1.4962740 | NO |
| 14 | 1.071 | 1.4652290 | NO |
| 15 | 1.067 | 1.4379890 | NO |
| 16 | 1.063 | 1.4138780 | NO |
| 17 | 1.059 | 1.3923700 | NO |
| 18 | 1.056 | 1.3730690 | NO |
| 19 | 1.053 | 1.3556360 | NO |
| 20 | 1.050 | 1.3398040 | NO |
| 21 | 1.048 | 1.3253690 | NO |
| 22 | 1.045 | 1.3121390 | NO |
| 23 | 1.043 | 1.2999750 | NO |
| 24 | 1.042 | 1.2887430 | NO |
| 25 | 1.040 | 1.2783470 | NO |
| 26 | 1.038 | 1.2686840 | NO |
| 27 | 1.037 | 1.2596920 | NO |
| 28 | 1.036 | 1.2512910 | NO |
| 29 | 1.034 | 1.2434240 | NO |
| 30 | 1.033 | 1.2360490 | NO |

TABLE 1B (Cont)

| N | DELTA - 1 + 1/N | MAXIMAL MOD OF THE ROOTS | STABILITY |
|---|---|---|---|
| 31 | 1.032 | 1.2291180 | NO |
| 32 | 1.031 | 1.2225870 | NO |
| 33 | 1.030 | 1.2164250 | NO |
| 34 | 1.029 | 1.2105970 | NO |
| 35 | 1.029 | 1.2050840 | NO |
| 36 | 1.028 | 1.1998570 | NO |
| 37 | 1.027 | 1.1948880 | NO |
| 38 | 1.026 | 1.1901720 | NO |
| 39 | 1.026 | 1.1856750 | NO |
| 40 | 1.025 | 1.1813930 | NO |
| 41 | 1.024 | 1.1773080 | NO |
| 42 | 1.024 | 1.1734020 | NO |
| 43 | 1.023 | 1.1696650 | NO |
| 44 | 1.023 | 1.1660900 | NO |
| 45 | 1.022 | 1.1626620 | NO |
| 46 | 1.022 | 1.1593790 | NO |
| 47 | 1.021 | 1.1562260 | NO |
| 48 | 1.021 | 1.1531930 | NO |
| 49 | 1.020 | 1.1502780 | NO |
| 50 | 1.020 | 1.1474740 | NO |

*NOTE: To give a more accurate picture, we also use the values* $\delta = N/50$, $N = 0, \ldots, 100$ *as in Table 1C.*

TABLE 1C. Stability VS. MAXIMUM MODULUS OF THE EIGENVALUES OF $A_1$

| N | DELTA = N/50 | MAXIMAL MOD OF THE ROOT | STABILITY |
|---|---|---|---|
| 0 | 0.0000 | 1.0040010 | NO |
| 1 | 0.0200 | 1.1561870 | NO |
| 2 | 0.0400 | 1.1873490 | NO |
| 3 | 0.0600 | 1.2053820 | NO |
| 4 | 0.0800 | 1.2170370 | NO |
| 5 | 0.1000 | 1.2247450 | NO |
| 6 | 0.1200 | 1.2296930 | NO |
| 7 | 0.1400 | 1.2325600 | NO |
| 8 | 0.1600 | 1.2337690 | NO |
| 9 | 0.1800 | 1.2336110 | NO |
| 10 | 0.2000 | 1.2325780 | NO |
| 11 | 0.2200 | 1.2299390 | NO |
| 12 | 0.2400 | 1.2266840 | NO |
| 13 | 0.2600 | 1.2226040 | NO |
| 14 | 0.2800 | 1.2177650 | NO |
| 15 | 0.3000 | 1.2122150 | NO |
| 16 | 0.3200 | 1.2060020 | NO |

TABLE 1C (Cont)

| N | DELTA = N/50 | MAXIMAL MOD OF THE ROOT | STABILITY |
|---|---|---|---|
| 17 | 0.3400 | 1.1991330 | NO |
| 18 | 0.3600 | 1.1916550 | NO |
| 19 | 0.3800 | 1.1835760 | NO |
| 20 | 0.4000 | 1.1749050 | NO |
| 21 | 0.4200 | 1.1656540 | NO |
| 22 | 0.4400 | 1.1558210 | NO |
| 23 | 0.4600 | 1.1454680 | NO |
| 24 | 0.4800 | 1.1344070 | NO |
| 25 | 0.5000 | 1.1227820 | NO |
| 26 | 0.5200 | 1.1105690 | NO |
| 27 | 0.5400 | 1.0977530 | NO |
| 28 | 0.5600 | 1.0842650 | NO |
| 29 | 0.5800 | 1.0700990 | NO |
| 30 | 0.6000 | 1.0552270 | NO |
| 31 | 0.6200 | 1.0396080 | NO |
| 32 | 0.6400 | 1.0232000 | NO |
| 33 | 0.6600 | 1.0059480 | NO |
| 34 | 0.6800 | 0.9877929 | YES |
| 35 | 0.7000 | 0.9686793 | YES |
| 36 | 0.7200 | 0.9484389 | YES |
| 37 | 0.7400 | 0.9270499 | YES |
| 38 | 0.7600 | 0.9043519 | YES |
| 39 | 0.7800 | 0.8801807 | YES |
| 40 | 0.8000 | 0.8543406 | YES |
| 41 | 0.8200 | 0.8265516 | YES |
| 42 | 0.8400 | 0.7964933 | YES |
| 43 | 0.8600 | 0.7638065 | YES |
| 44 | 0.8800 | 0.7275256 | YES |
| 45 | 0.9000 | 0.6870258 | YES |
| 46 | 0.9200 | 0.6407076 | YES |
| 47 | 0.9400 | 0.5859586 | YES |
| 48 | 0.9600 | 0.5176454 | YES |
| 49 | 0.9800 | 0.8165164 | YES |
| 50 | 1.0000 | 0.9999986 | NO |
| 51 | 1.0200 | 1.1474740 | NO |
| 52 | 1.0400 | 1.2783470 | NO |
| 53 | 1.0600 | 1.3992770 | NO |
| 54 | 1.0800 | 1.5134960 | NO |
| 55 | 1.1000 | 1.6228290 | NO |
| 56 | 1.1200 | 1.7284450 | NO |
| 57 | 1.1400 | 1.8311170 | NO |
| 58 | 1.1600 | 1.9314120 | NO |
| 59 | 1.1800 | 2.0297310 | NO |
| 60 | 1.2000 | 2.1263980 | NO |
| 61 | 1.2200 | 2.2216460 | NO |
| 62 | 1.2400 | 2.3156780 | NO |
| 63 | 1.2600 | 2.4086440 | NO |
| 64 | 1.2800 | 2.5006810 | NO |
| 65 | 1.3000 | 2.5918890 | NO |

TABLE 1C (Cont)

| N | DELTA = N/50 | MAXIMAL MOD OF THE ROOT | STABILITY |
|---|---|---|---|
| 66 | 1.3200 | 2.6823650 | NO |
| 67 | 1.3400 | 2.7721790 | NO |
| 68 | 1.3600 | 2.8614030 | NO |
| 69 | 1.3800 | 2.9500870 | NO |
| 70 | 1.4000 | 3.0382880 | NO |
| 71 | 1.4200 | 3.1260400 | NO |
| 72 | 1.4400 | 3.2133900 | NO |
| 73 | 1.4600 | 3.3003620 | NO |
| 74 | 1.4800 | 3.3869950 | NO |
| 75 | 1.5000 | 3.4733050 | NO |
| 76 | 1.5200 | 3.5593280 | NO |
| 77 | 1.5400 | 3.6450760 | NO |
| 78 | 1.5600 | 3.7305680 | NO |
| 79 | 1.5800 | 3.8158280 | NO |
| 80 | 1.6000 | 3.9008630 | NO |
| 81 | 1.6200 | 3.9856960 | NO |
| 82 | 1.6400 | 4.0703320 | NO |
| 83 | 1.6600 | 4.1547900 | NO |
| 84 | 1.6800 | 4.2390740 | NO |
| 85 | 1.7000 | 4.3232030 | NO |
| 86 | 1.7200 | 4.4071750 | NO |
| 87 | 1.7400 | 4.4910080 | NO |
| 88 | 1.7600 | 4.5747040 | NO |
| 89 | 1.7800 | 4.6582750 | NO |
| 90 | 1.8000 | 11.1541500 | NO |
| 91 | 1.8200 | 4.7706570 | NO |
| 92 | 1.8400 | 4.9082810 | NO |
| 93 | 1.8600 | 4.9914050 | NO |
| 94 | 1.8800 | 5.0744260 | NO |
| 95 | 1.9000 | 4.6608080 | NO |
| 96 | 1.9200 | 5.2401970 | NO |
| 97 | 1.9400 | 5.3229540 | NO |
| 98 | 1.9600 | 5.4056270 | NO |
| 99 | 1.9800 | 5.4882250 | NO |

The transfer functions $\underline{H}_\delta(z) := [H_1(z), H_2(z), H_3(z)]^T$ can also be obtained by substituting $\underline{w} = [\delta, \frac{2\delta}{h}, \frac{2\delta}{h^2}]^T$ into (2.8) - (2.10). We have

$$H_1(z) = \frac{\delta + \delta z^{-1}}{1 + (4\delta - 3)z^{-1} + 3(1-\delta)z^{-2} + (\delta - 1)z^{-3}}, \tag{3.7}$$

$$H_2(z) = \frac{\frac{2\delta}{h}(1 - z^{-1})}{1 + (4\delta - 3)z^{-1} + 3(1-\delta)z^{-2} + (\delta - 1)z^{-3}}, \tag{3.8}$$

and

$$H_3(z) = \frac{\frac{2\delta}{h^2}(1 - z^{-1})^2}{1 + (4\delta - 3)z^{-1} + 3(1-\delta)z^{-2} + (\delta - 1)z^{-3}}, \tag{3.9}$$

Also, by applying Theorem 2.2, we can uncouple the system (3.3) to yield:

$$y_p = -(4\delta - 3)y_{p-1} - 3(1-\delta)y_{p-2} + (1-\delta)y_{p-3} + \delta x_p + \delta x_{p-1}, \tag{3.10}$$

$$y'_p = -(4\delta - 3)y'_{p-1} - 3(1-\delta)y'_{p-2} + (1-\delta)y'_{p-3} + \frac{2\delta}{h}x_p - \frac{2\delta}{h}x_{p-1} \tag{3.11}$$

and

$$y''_p = -(4\delta - 3)y''_{p-1} - 3(1-\delta)y''_{p-2} + (1-\delta)y''_{p-3} + \frac{2\delta}{h^2}x_p - \frac{4\delta}{h^2}x_{p-1} + \frac{2\delta}{h^2}x_{p-2}, \tag{3.12}$$

for $p = 0, 1, 2, \ldots$ with initial conditions $x_{-1} = x_{-2} = 0$ and $y_{-2} = y_{-3} = 0$ and $\underline{y}_{-1} = [y_{-1}, y'_{-1}, y''_{-1}]^T$ preassigned. Equations (3.10) - (3.12) enable the user to plot the output response $\underline{y}_p = [y_p, y'_p, y''_p]^T$ in three different graphs, where the graph of $\{y_p\}$ shows how the filter smooths the input signal $\{x_p\}$, $p = 0, 1, \ldots$ . It is also interesting to study how close $\{y_p\}$ is to $\{x_p\}$ when $\delta$ is close to but different from 1. This can be done using the transfer function $H_1(z)$ given in (3.7) as in the following. Since the filter is stable for $\frac{2}{3} < \delta < 1$, we always pick such values of $\delta$. For simplicity, we set

$$\varepsilon = 1 - \delta > 0$$

so that we have

$$1 - H_1(z) = \varepsilon \frac{(1 - z^{-1})^3}{1 + (1-4\varepsilon)z^{-1} + 3\varepsilon z^{-2} - \varepsilon z^{-3}} . \tag{3.13}$$

By applying Parseval's identity, we have

$$\begin{aligned}\sum_{p=-1}^{\infty} |x_p - y_p|^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(e^{-i\omega}) - Y_1(e^{-i\omega})|^2 dw \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |1 - H_1(e^{-i\omega})|^2 |X(\omega)|^2 \, d\omega \end{aligned} \tag{3.14}$$

where we have used the common notation $X(\omega) = X(e^{-i\omega})$ and $x_{-1} = 0$. Hence, from (3.13), we have

$$|y_{-1}|^2 + \sum_{p=0}^{\infty} |x_p - y_p|^2 = \frac{\varepsilon^2}{2\pi}\int_{-\pi}^{\pi} \left| \frac{(1 - e^{i\omega})^3}{1 + (1-4\varepsilon)e^{i\omega} + 3\varepsilon e^{i2\omega} - \varepsilon e^{i3\omega}} \right|^2 |X(\omega)|^2 d\omega$$

$$= \frac{4\varepsilon^2}{\pi}\int_{-\pi}^{\pi} \frac{(1 - \cos\omega)^3}{(1 + \rho_\varepsilon \cos\omega + 3\varepsilon\cos 2\omega - \varepsilon\cos 3\omega)^2 + (\rho_\varepsilon \sin\omega + 3\varepsilon\sin 2\omega - \varepsilon\sin 3\omega)^2} |X(\omega)|^2 \, d\omega, \tag{3.15}$$

where $\rho_\varepsilon := 1 - 4\varepsilon$ .

The last expression allows us to design Shepherd's parabolic spline predictor-corrector digital filter in an optimal way. Depending on the input spectrum $X(\omega)$, one can pick $\varepsilon = 1 - \delta$ (numerically) such that $0 < \varepsilon < 1/3$ and such that the last expression above is minimized. In particular, if we have faith on the data $\{x_p\}$, then we can pick $\varepsilon > 0$ very close to zero. This gives

$$|y_{-1}|^2 + \sum_{p=0}^{\infty} |x_p - y_p|^2 \doteq \frac{4\varepsilon^2}{\pi} \int_{-\pi}^{\pi} \frac{(1-\cos\omega)^3}{1+2\rho_\varepsilon \cos\omega + \rho_\varepsilon^2} |X(\omega)|^2 \, d\omega \quad (3.16)$$

where $\rho_\varepsilon = 1 - 4\varepsilon$ as above. This integral can be used to design the filter efficiently.

## 4. THE α-β-γ TRACKERS

The α-β-γ tracking equations discussed by Steelman [5] can be obtained from (1.3) - (1.4) by putting $\underline{w} = [\alpha, \beta/h, \gamma/h^2]^T$. This shows that each α-β-γ tracker is a third order predictor-corrector filter, and conversely (e.g. the filter of section 3 is an α-β-γ tracker with $\beta = \gamma = 2\delta$). From (2.8) - (2.10), the transfer functions of this filter can be found to be

$$H_1(z) = \frac{\alpha + (-2\alpha+\beta+\frac{1}{2}\gamma)z^{-1} + (\alpha-\beta+\frac{1}{2}\gamma)z^{-2}}{1 + (\alpha+\beta+\frac{1}{2}\gamma-3)z^{-1} + (-2\alpha-\beta+\frac{1}{2}\gamma+3)z^{-2} + (\alpha-1)z^{-3}} \quad (4.1)$$

$$H_2(z) = \frac{\frac{\beta}{h} + (-2\frac{\beta}{h} + \frac{\gamma}{h})z^{-1} + (\frac{\beta}{h} - \frac{\gamma}{h})z^{-2}}{1 + (\alpha+\beta+\frac{1}{2}\gamma-3)z^{-1} + (-2\alpha-\beta+\frac{1}{2}\gamma+3)z^{-2} + (\alpha-1)z^{-3}} \quad (4.2)$$

and

$$H_3(z) = \frac{\frac{\gamma}{h^2} - 2\frac{\gamma}{h^2}z^{-1} + \frac{\gamma}{h^2}z^{-2}}{1 + (\alpha+\beta+\frac{1}{2}\gamma-3)z^{-1} + (-2\alpha-\beta+\frac{1}{2}\gamma+3)z^{-2} + (\alpha-1)z^{-3}} \quad (4.3)$$

These equations were also obtained in [5] in a different form. However, using the formulation in (4.1) - (4.3), we can immediately uncouple the filter as in section 2, yielding

$$\begin{aligned} y_p = & -(\alpha+\beta+\tfrac{1}{2}\gamma-3)y_{p-1} + (2\alpha+\beta-\tfrac{1}{2}\gamma-3)y_{p-2} - (\alpha-1)y_{p-3} \\ & + \alpha x_p - (2\alpha-\beta-\tfrac{1}{2}\gamma)x_{p-1} + (\alpha-\beta+\tfrac{1}{2}\gamma)x_{p-2}\,, \end{aligned} \quad (4.4)$$

$$y_p' = -(\alpha+\beta+\tfrac{1}{2}\gamma-3)y_{p-1}' + (2\alpha+\beta-\tfrac{1}{2}\gamma-3)y_{p-2}' - (\alpha-1)y_{p-3}'$$
$$+ \frac{\beta}{h}x_p - \left(\frac{2\beta}{h} - \frac{\gamma}{h}\right)x_{p-1} + \left(\frac{\beta}{h} - \frac{\gamma}{h}\right)x_{p-2}\,, \tag{4.5}$$

and

$$y_p'' = -(\alpha+\beta+\tfrac{1}{2}\gamma-3)y_{p-1}'' + (2\alpha+\beta-\tfrac{1}{2}\gamma-3)y_{p-2}'' - (\alpha-1)y_{p-3}''$$
$$+ \frac{\gamma}{h^2}x_p - 2\frac{\gamma}{h^2}x_{p-1} + \frac{\gamma}{h^2}x_{p-2}\,. \tag{4.6}$$

where $p = 0, 1, 2, \ldots$ , $y_p = y_p' = y_p'' = 0$ if $p < -1$ and $x_{-1} = x_{-2} = 0$.

The case where $\alpha = 1 - \theta^3$, $\beta = \frac{3}{2}(1-\theta)^2(1+\theta)$, and $\gamma = (1-\theta)^3$ is called a "fading memory polynomial filter of degree 2" by Morrison (cf. [5]). The uncoupled recursive filters (4.4) - (4.6) can be simplified to be

$$y_p = 3\theta y_{p-1} - 3\theta^2 y_{p-2} + \theta^3 y_{p-3} + (1-\theta^3)x_p$$
$$- 3\theta(1-\theta^2)x_{p-1} + 3\theta^2(1-\theta)x_{p-2}\,, \tag{4.7}$$

$$y_p' = 3\theta y_{p-1}' - 3\theta^2 y_{p-2}' + \theta^3 y_{p-3}' + \frac{3}{2h}(1-\theta)^2(1+\theta)x_p$$
$$- \frac{2}{h}(1-\theta)^2(1+2\theta)x_{p-1} + \frac{1}{2h}(1-\theta)^2(1+5\theta)x_{p-2}\,, \tag{4.8}$$

and

$$y_p'' = 3\theta y_{p-1}'' - 3\theta^2 y_{p-2}'' + \theta^3 y_{p-3}'' + \frac{(1-\theta)^3}{h^2}(x_p - 2x_{p-1} + x_{p-2}), \tag{4.9}$$

for $p = 0, 1, \ldots$ with $y_p = y_p' = y_p'' = 0$ for $p < -1$ and $x_{-1} = x_{-2} = 0$.

Note that the feed-back coefficients are particularly simple. The stability of this filter is also particularly easy to check. In fact, it is stable if and only if $|\theta| < 1$.

As another special case of the $\alpha$-$\beta$-$\gamma$ filter, let us set $\gamma = 0$. This is the so-called $\alpha$-$\beta$ tracker, studied in WSMR by Gonzales in 1968 (cf. [2]). In this case, the factor $1 - z^{-1}$ can be cancelled in each of the expressions in (4.1), (4.2), and (4.3); and the transfer functions of the $\alpha$-$\beta$ tracker simply become

$$H_1(z) = \frac{\alpha - (\alpha-\beta)z^{-1}}{1 + (\alpha+\beta-2)z^{-1} - (\alpha-1)z^{-2}}, \tag{4.10}$$

$$H_2(z) = \frac{\frac{\beta}{h} - \frac{\beta}{h} z^{-1}}{1 + (\alpha+\beta-2)z^{-1} - (\alpha-1)z^{-2}}, \tag{4.11}$$

and

$$H_3(z) \equiv 0. \tag{4.12}$$

Since the denominator is a quadratic polynomial, we know immediately that this $\alpha$-$\beta$ filter is stable if and only if $\Lambda_{\alpha,\beta} < 1$ where

$$\Lambda_{\alpha,\beta} := \max\left[\left|\frac{\alpha+\beta}{2} - 1 - \sqrt{\left(\frac{\alpha+\beta}{2}\right)^2 - \beta}\right|, \left|\frac{\alpha+\beta}{2} - 1 + \sqrt{\left(\frac{\alpha+\beta}{2}\right)^2 - \beta}\right|\right] \tag{4.13}$$

Hence, it follows easily that for $\beta \geq [(\alpha + \beta)/2]^2$, we have $\Lambda_{\alpha,\beta} < 1$ if and only if $\alpha > 0$. That is, a stability condition for the $\alpha$-$\beta$ tracker is:

$$\alpha > 0, \quad (\alpha + \beta)^2 \leq 4\beta . \tag{4.14}$$

Another sufficient condition for stability of the $\alpha$-$\beta$ filters is

$$\beta > 0, \quad \alpha + \beta \leq 2. \tag{4.15}$$

Of course, the other sufficient condition for stability

$$\alpha > 0, \quad \beta > 0, \quad 2\alpha + \beta < 4, \tag{4.16}$$

which also follows easily, was obtained by Gonzales (cf. [2,5]). Using an inverse z-transform as in section 2, the $\alpha$-$\beta$ tracker equations can be uncoupled into the form:

$$y_p = -(\alpha+\beta-2)y_{p-1} + (\alpha-1)y_{p-2} + \alpha x_p - (\alpha-\beta)x_{p-1}, \tag{4.17}$$

$$y'_p = -(\alpha+\beta-2)y'_{p-1} + (\alpha-1)y'_{p-2} + \frac{\beta}{h}(x_p - x_{p-1}), \tag{4.18}$$

and $y''_p = 0$, $p = 0, 1, \ldots,$ where $y_p = y'_p = 0$ if $p < -1$ and $x_p = 0$ if $p < 0$. The condition (4.15) is very useful. It says that if the first feed-back coefficient is nonnegative and $\beta > 0$, then the $\alpha$-$\beta$ filter (4.17) - (4.19) is always stable.

Finally, let us discuss McCool's QD filter [cf. 3], namely:

$$\alpha = \frac{60M^2}{10M^3 + 33M^2 + 23M - 6}, \tag{4.19}$$

$$\beta = \frac{2\alpha}{M} \quad \text{and} \quad \gamma = \frac{2\alpha}{M^2}, \tag{4.20}$$

where $M$ is a natural number. The transfer functions of this filter can be obtained by substituting (4.20) into (4.1) - (4.3) and the uncoupled recursive input-output relationship can be obtained from (4.4) - (4.6). We now study the stability of the QD filter. Set

$$f(z) = z^3 + [(M+1)^2 \rho_M - 3]z^2 + [(-2M^2 - 2M + 1)\rho_M + 3]z + (M^2 \rho_M - 1)$$

with $\rho_M = 60/(10M^3 + 33M^2 + 23M - 6)$. If $\lambda_1$, $\lambda_2$, and $\lambda_3$ are the roots of $f(z) = 0$ and $\Lambda_M = \max(|\lambda_1|, |\lambda_2|, |\lambda_3|)$, then by Corollary 2.1, since $w_3 = \gamma/h^2 = 2\rho_M \neq 0$, we note that the QD digital filter is stable if and only if $\Lambda_M < 1$. The following table indicates the stability of this filter for $M = 0, \pm 1, \pm 2, \ldots, \pm 20$ .

TABLE 2. STABILITY FOR McCOOL'S QD FILTER

| M | RO = $P_M$ | MAXIMAL MOD OF THE ROOT | STABILITY |
|---|---|---|---|
| -20 | -0.0009 | 1.2024780 | NO |
| -19 | -0.0011 | 1.2146490 | NO |
| -18 | -0.0012 | 1.2282810 | NO |
| -17 | -0.0015 | 1.2437600 | NO |
| -16 | -0.0018 | 1.2615480 | NO |
| -15 | -0.0022 | 1.2820790 | NO |
| -14 | -0.0028 | 1.3061450 | NO |
| -13 | -0.0036 | 1.3346260 | NO |
| -12 | -0.0047 | 1.3689580 | NO |
| -11 | -0.0063 | 1.4111290 | NO |
| -10 | -0.0087 | 1.4641570 | NO |
| -9 | -0.0124 | 1.5328290 | NO |
| -8 | -0.0188 | 1.6252010 | NO |
| -7 | -0.0303 | 1.7560380 | NO |
| -6 | -0.0538 | 1.9554370 | NO |
| -5 | -0.1099 | 2.2953960 | NO |
| -4 | -0.2857 | 3.0000000 | NO |
| -3 | -1.2500 | 5.2570590 | NO |
| -2* | | | |
| -1 | -10.0000 | 4.8910190 | NO |
| 0 | -10.0000 | 13.5226500 | NO |
| 1 | 1.0000 | 0.9999990 | YES |
| 2 | 0.2381 | 0.5061458 | YES |
| 3 | 0.0952 | 0.6546553 | YES |
| 4 | 0.0478 | 0.7329382 | YES |
| 5 | 0.0275 | 0.7819132 | YES |
| 6 | 0.0172 | 0.8155725 | YES |
| 7 | 0.0115 | 0.8401833 | YES |
| 8 | 0.0081 | 0.8589627 | YES |
| 9 | 0.0059 | 0.8737922 | YES |
| 10 | 0.0044 | 0.8857723 | YES |
| 11 | 0.0034 | 0.8956649 | YES |
| 12 | 0.0027 | 0.9040021 | YES |
| 13 | 0.0022 | 0.9110962 | YES |
| 14 | 0.0018 | 0.9172200 | YES |
| 15 | 0.0014 | 0.9225416 | YES |
| 16 | 0.0012 | 0.9272274 | YES |
| 17 | 0.0010 | 0.9313622 | YES |
| 18 | 0.0009 | 0.9350671 | YES |
| 19 | 0.0007 | 0.9383648 | YES |
| 20 | 0.0006 | 0.9413764 | YES |

*Coefficients of the polynomials are undefined; division by zero.

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